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Generalized Poisson Algebras and Hamiltonian Dynamics

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Hamiltonian dynamics can be formulated entirely in terms of a Poisson manifold, that is, one for which the algebra of smooth functions is a Poisson algebra. The latter is a commutative associative algebra A together with a skew-symmetric bracket which is a derivation on A. It is shown that a Poisson algebra can be generalized by replacing A by algebras which do not necessarily commute. These allow for algebraic generalizations of Hamiltonian dynamics in both classical and quantum forms. Particular examples are models of classical and quantum electrons.

1. INTRODUCTION

Poisson algebras arose from the generalization of symplectic manifolds to Poisson algebras (Bhaskara and Viswanath, 1988; Libermann and Marle, 1987; Guillemin and Sternberg, 1980). Indeed, a Poisson manifold can be defined to be a manifold M for which the algebra of smooth functions on M has the structure of a Poisson algebra.

The basic structures of Hamiltonian dynamics are the Hamiltonian vector fields and the associated equations governing the flow of these vector fields, namely Hamilton's equations. Although it is usual to define these constructions on a symplectic manifold, the latter is not essential. It is possible to formulate these structures entirely in terms of Poisson manifolds (Olver, 1986) and consequently in terms of the associated Poisson algebra.

In this paper the notion of a Poisson algebra is generalized by replacing the commutative algebra of functions defined on a manifold with certain algebras which are not necessarily commutative. Such a procedure fits in with the basic idea of noncommutative geometry (Connes, 1986; Dubois-Violette *et al.*, 1989). These algebras allow for the formulation of an analog

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of Hamiltonian mechanics entirely in algebraic terms. In this way it is possible to reveal an essential unity among different dynamical structures such as classical and quantum dynamics in general, and in particular models of classical electrons and their quantum counterparts.

The organization of this work is as follows. In Section 2 we present a brief description of Poisson algebras and their relationship with classical Hamiltonian dynamics. Generalized Poisson algebras and Hamiltonian dynamical structures are introduced in Section 3. In Section 4 algebraic analogs of classical and quantum dynamics are discussed. Algebraic dynamical models of classical electrons are considered in Section 5, while in Section 6 quantum equivalents of these models are discussed.

2. POISSON ALGEBRAS AND HAMILTONIAN DYNAMICS

A Poisson algebra can be defined as follows: Let A be a commutative associative algebra over $\mathbb{R}$ with unit. A *Poisson bracket* $\{,\}$ on A is a map: $A\times A\to A$ satisfying

(a) bilinearity (2.1)

(b) $\{f, g\}=-\{g, f\}$ (2.2)

(c) $\{f, gh\}=\{f, g\}h+g\{f, h\}$ (2.3)
i.e., the bracket is a derivation on A

(d) $\{\{f, g\}, h\}+\{\{g, h\}, f\}+\{\{h, f\}, g\}=0$ (2.4)
for f, g, $h\in A$

This algebra is denoted by the pair $(A, \{,\})$. We can relate this to Hamiltonian dynamics. Let M be a manifold and let A represent the commutative algebra of C^∞ functions on M. Then M and the Poisson algebra $(A, \{,\})$ constitute a *Poisson manifold.* A *Hamiltonian vector field* associated with a function $h\in A$ is given by V_h where

$$V_h(g)=\{g, h\} \tag{2.5}$$

for $g\in A$. In the case where M is the space $\mathbb{R}^{2n}$ it is always possible to find canonical coordinates (q^j, p_j), $j=1, 2, \ldots, n$, such that for any $f, g\in A$,

$$\{f, g\}=\sum_{j=1}^{n}\left(\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j}-\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j}\right) \tag{2.6}$$

The integral curves $(q^j(t), p_j(t))$ of a Hamiltonian vector field V_h satisfy Hamilton's equations

$$\frac{dq^j}{dt}=\{q^j, h\}=\frac{\partial h}{\partial p_j}; \qquad \frac{dp_j}{dt}=\{p_j, h\}=-\frac{\partial h}{\partial q^j} \tag{2.7}$$

and it follows for any $f \in A$ which is not an explicit function of t that

$$\frac{df}{dt} = \{f, h\} \tag{2.8}$$

3. GENERALIZED POISSON ALGEBRAS AND HAMILTONIAN DYNAMICAL STRUCTURES

We shall now generalize the (commutative) Poisson algebra to the case where A is replaced by an associative, but not necessarily commutative, algebra $\mathscr{A}$ and denote the generalized Poisson algebra by $(\mathscr{A}, \{,\})$. Let

$$x_J = (q_j, p_j, z_\mu)$$

where $j = 1, 2, \ldots, n$; $\mu = 1, 2, \ldots, m$; $J = 1, 2, \ldots, 2n+m$; and assume that x_J are generators of $\mathscr{A}$.

We can now construct the equivalent of Hamiltonian dynamics on $(\mathscr{A}, \{,\})$. Let $\mathrm{Der}(\mathscr{A})$ denote the space of all derivations on $\mathscr{A}$ and let

$$\left\{\frac{\partial}{\partial x_J} = \frac{\partial}{\partial q_j}, \frac{\partial}{\partial p_j}, \frac{\partial}{\partial z_\mu}\right\}$$

be a subset of $\mathrm{Der}(\mathscr{A})$ satisfying

$$\begin{gathered}\frac{\partial q_j}{\partial q_k} = \delta^j_k = \frac{\partial p_j}{\partial p_k}; \qquad \frac{\partial q_j}{\partial p_k} = 0 = \frac{\partial p_j}{\partial q_k} \\ \frac{\partial q_j}{\partial z_\mu} = \frac{\partial p_j}{\partial z_\mu} = 0 = \frac{\partial z_\mu}{\partial q_j} = \frac{\partial z_\mu}{\partial p_j}\end{gathered} \tag{3.1}$$

while the properties of $\partial z_\nu / \partial z_\mu$ are as yet unspecified. These derivations will be referred to as the *canonical derivations.*

A *Poisson bracket* is defined on $\mathscr{A}$ as a map: $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ which satisfies all the properties (2.1)-(2.4) with A replaced by $\mathscr{A}$. Then the pair $(\mathscr{A}, \{,\})$ is the *Poisson algebra* defined on $\mathscr{A}$.

The algebraic equivalent of a Hamiltonian vector field is the *Hamiltonian derivation* $V_h \in \mathrm{Der}(\mathscr{A})$ associated with $h \in \mathscr{A}$, satisfying

$$V_h(x_J) = \{x_J, h\}$$

such that

$$V_h(q_j) = \frac{\partial h}{\partial p_j}; \qquad V_h(p_j) = -\frac{\partial h}{\partial q_j}; \qquad V_h(z_\mu) = \frac{\partial h}{\partial z_\mu} \tag{3.2}$$

In particular,

$$\{q_j, q_k\} = \{p_j, p_k\} = 0; \qquad \{q_j, p_k\} = \delta_{jk} \tag{3.3}$$

while $\{z_\mu, z_\nu\}$ are as yet unspecified. We shall now show that for any $f \in \mathscr{A}$

$$V_h(f) = \{f, h\} \tag{3.4}$$

Since $V_h \in \mathrm{Der}(\mathscr{A})$, it follows that

$$\begin{aligned} V_h(x_J x_K) &= V_h(x_J) x_K + x_J V_h(x_K) \\ &= \{x_J, h\} x_K + x_J \{x_K, h\} \\ &= \{x_J x_K, h\} \qquad \text{by (2.3)} \end{aligned}$$

Now $f \in \mathscr{A}$ consists of sums and products of terms such as $x_J x_K$ and so it is sufficient to show that if (3.4) holds for two elements $f_1, f_2 \in \mathscr{A}$, then it holds for $f_1 + f_2$ and $f_1 f_2$. The first follows since V_h is linear, while the second follows from the derivation property of V_h.

In analogy with Hamilton's equations, we define a *Hamiltonian dynamical structure* on $(\mathscr{A}, \{,\})$. We first require the algebraic equivalent of a curve on a manifold and define this in the following way. If $f \in \mathscr{A}$, let φ_f be a continuous map from an interval $I \subset \mathbb{R}^+ \to \mathscr{A}$ such that if $t_i \in I$, then

$$\varphi_f\colon \quad t_i \to f(t_i) \in \mathscr{A} \tag{3.5}$$

The set $\{f(t_i)\}$ for all $t_i \in I$ will be denoted by $f(t)$ and we shall refer to $f(t)$ as a *function* of t.

Next introduce a map $h_\varepsilon\colon f(t) \to f(t)$, where $h \in \mathscr{A}$, which we shall call an *infinitesimal t-canonical transformation.* Let $\varepsilon, \delta, \ldots \in \mathbb{R}$ be such that products of the order of two or greater can be neglected. Let t_j be an element of I and assume that $(t_i + \varepsilon) \in I$, so that $f(t_i), f(t_i + \varepsilon) \in f(t)$. Then we define

$$h_\varepsilon(f(t_i)) = f(t_i) + \varepsilon\{f(t_i), h\} \tag{3.6}$$

It can then be shown that:

(a) $h_\varepsilon[h_\delta f(t_i)] = h_{\varepsilon+\delta} f(t_i)$.
(b) $h_0 f(t_i) = f(t_i)$.
(c) h_ε preserves the relations (3.3).

This map allows us to introduce the notion of a *t-derivative* via

$$\frac{df}{dt} = \lim_{\varepsilon \to 0} \frac{h_\varepsilon[f(t)] - f(t)}{\varepsilon} = \{f(t), h\} \tag{3.7}$$

Hamiltonian dynamical structure is defined on $(\mathscr{A}, \{,\})$ if $h \in \mathscr{A}$ exists such that

$$\frac{dx_J}{dt} = \{x_J, h\} \tag{3.8}$$

and h is then called a *Hamiltonian*. This result can be extended to

$$\frac{df}{dt} = \{f, h\} \tag{3.9}$$

for any $f \in \mathscr{A}$ not containing t explicitly.

4. ALGEBRAIC CLASSICAL AND QUANTUM DYNAMICS

An algebraic model of classical Hamiltonian dynamics can be formulated in the case where $\mathscr{A}$ is a commutative algebra over $\mathbb{R}$ with generators

$$x_J = (q_j, p_j), \qquad j = 1, \ldots, n, \quad J = 1, \ldots, 2n \tag{4.1}$$

and a unit element $1 \in \mathbb{R}$.

Denote this algebra by C. Define the canonical derivations to be $(\partial/\partial q_j, \partial/\partial p_j)$ satisfying (3.1); then a Poisson bracket on C is given by

$$\{f, g\}_C = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right) \tag{4.2}$$

for $f, g \in C$. Every element of C is assumed to be a function of t as defined in Section 3 and a Hamiltonian dynamical structure on C can be constructed on the Poisson algebra $(C, \{,\}_C)$ as defined above.

If $\mathscr{A}$ is replaced by an nth Weyl algebra $\mathscr{W}$ (McConnell and Robson, 1987), then we obtain an algebraic version of quantum dynamics. By *quantum dynamics* we refer only to the Heisenberg picture of quantum mechanics restricted specifically to Heisenberg's equation of motion. The algebra $\mathscr{W}$ over $\mathbb{C}$ has generators $\tilde{x}_J = (\tilde{q}_j, \tilde{p}_j)$ which satisfy the relations

$$[\tilde{p}_i, \tilde{p}_j] = 0 = [\tilde{q}_i, \tilde{q}_j]; \qquad [\tilde{q}_i, \tilde{p}_i] = \delta_{ij} I \tag{4.3}$$

where I is the unit element. Let $\tilde{f}$ be an element of $\mathscr{W}$ and define the derivation $\partial/\partial \tilde{q}^j$ by

$$\frac{\partial \tilde{f}}{\partial \tilde{q}_j} = \lim_{\varepsilon \to 0} \frac{\tilde{f}(\tilde{q}_j + \varepsilon I, \tilde{p}_j) - \tilde{f}(\tilde{q}_j, \tilde{p}_j)}{\varepsilon} \tag{4.4}$$

and $\partial/\partial \tilde{p}_j$ similarly. The commutator [,] satisfies all the properties (2.1)-(2.4) and so qualifies as a Poisson bracket. The pair $(\mathscr{W}, [,])$ form a noncommutative Poisson algebra. It has been shown (Sherry, 1989*b*) that from the definition of (4.4) we get

$$\frac{\partial}{\partial \tilde{q}_j} = -[\tilde{p}_j, \tilde{f}]; \qquad \frac{\partial \tilde{f}}{\partial \tilde{p}_j} = [\tilde{q}_j, \tilde{f}] \tag{4.5}$$

If every element of $\mathscr{W}$ is a function of t and if $\tilde{h}$ is a Hamiltonian, then

$$\frac{d\tilde{q}_j}{dt}=[\tilde{q}_j, \tilde{h}]; \qquad \frac{d\tilde{p}_j}{dt}=[\tilde{p}_j, \tilde{h}] \tag{4.6}$$

and, in general, if $\tilde{f}$ is not an explicit function of t, we obtain

$$\frac{d\tilde{f}}{dt}=[\tilde{f}, \tilde{h}] \tag{4.7}$$

which is the equivalent of Heisenberg's equation.

5. POISSON ALGEBRAS AND DYNAMICAL MODELS OF CLASSICAL ELECTRONS

In previous work (Sherry, 1989*a*–*c*) algebraic models of classical electrons were introduced. In this section we reformulate these in terms of generalized Poisson algebras. In these models the notion of classical spin is provided by bivectors of suitable Grassmann algebras.

Although other models of classical (or pseudoclassical) spinning particles based on Grassmann algebras exist (Berezin and Marinov, 1977; Casalbuoni, 1976*a,b*; Barducci *et al.*, 1976; Gomis *et al.*, 1955), none of these provide physical interpretations of the Grassmann variables. The formalism presented here allows for direct physical and geometrical interpretations formally similar to equivalent variables in quantum theory.

The relevant Poisson algebra is defined on the tensor product of a Grassmann algebra and C denoted by

$$\mathscr{D}=\mathscr{G}\otimes C$$

Here $\mathscr{D}$ has as generators

$$x_J=(q_j, p_j, s_j) \tag{5.1}$$

where

$$[q_j, p_j]=[q_j, q_k]=[p_j, p_k]=0 \tag{5.2}$$

while

$$s_j s_k+s_k s_j=0 \tag{5.3}$$

and in addition

$$[q_j, s_k]=0=[p_j, s_k] \tag{5.4}$$

The s_j are generators of a Grassmann algebra $\mathscr{G}$ over $\mathbb{R}$, with unit element $1\in\mathbb{R}$. Here $\mathscr{G}$ is a Z_2 graded, or superalgebra, of the form

$$\mathscr{G}=\mathscr{G}_0\otimes\mathscr{G}_1 \tag{5.5}$$

such that if

$$a_i \in \mathscr{G}_i, \quad b_j \in \mathscr{G}_j \qquad \text{then} \quad a_i b_j \in \mathscr{G}_{i+j} \qquad \text{where} \quad i, j, i+j = 0, 1$$

and

$$a_i b_j = (-1)^{(i+j)} b_j a_i \tag{5.6}$$

In particular, if $a_0, b_0 \in \mathscr{G}_0$, then $a_0 b_0 = b_0 a_0$ and such elements are said to be homogeneous of degree zero, denoted by $(a_0) = 0 = (b_0)$. Elements $a_1, b_1 \in \mathscr{G}_1$ satisfy $a_1 b_1 = -b_1 a_1$ and we write $(a_1) = (b_1) = 1$.

Now since every element of C commutes with every element of $\mathscr{G}$, we can write

$$\mathscr{D} = \mathscr{D}_0 \otimes \mathscr{D}_1$$

where every element of C is homogeneous of degree zero. Associated with the generators of (5.1) we define canonical derivations $(\partial/\partial q_j, \partial/\partial p_j, \partial/\partial s_j)$ satisfying (3.1) with z_μ identified with s_j and also $\partial s_j/\partial s_k = \delta_{jk}$. The following additional results also hold:

$$\frac{\partial}{\partial s_j}\frac{\partial}{\partial s_k} + \frac{\partial}{\partial s_k}\frac{\partial}{\partial s_j} = 0 \tag{5.7}$$

$$s_j \frac{\partial}{\partial s_k} + \frac{\partial}{\partial s_k} s_j = \delta_{jk} \tag{5.8}$$

and

$$\frac{\partial}{\partial s_j} s_{k_1} \cdots s_{k_m} = \sum_{l=1}^{m} (-1)^{k_l - 1} \delta_{k_l i} s_{k_1} \cdots \hat{s}_{k_l} \cdots s_{k_m} \tag{5.9}$$

We require some further notation. Let $\Gamma, \Lambda \in \mathscr{G}$ be of the form

$$\Gamma = \begin{cases} s_{k_1} \cdots s_{k_m} & 1 \le k_1 < k_2 \cdots < k_m \le n \\ \text{or} & \\ \text{unit element } 1 \in \mathscr{G} & \end{cases}$$

Then $\Gamma \cap \Lambda$ denotes the set of s_j is common to both Γ and Λ. As a result, since any element $F \in \mathscr{D}$ can be written in the form Γf, where $f \in C$, this notation can be extended to $\mathscr{D}$ as well.

A Poisson bracket $\{,\}_{\mathscr{D}}$ is defined on $\mathscr{D}$ as follows:

$$\{F, G\}_{\mathscr{D}} = \begin{cases} g^{jk}\left(\dfrac{\partial F}{\partial q_j}\dfrac{\partial G}{\partial p_j} - \dfrac{\partial F}{\partial p_j}\dfrac{\partial G}{\partial q_j}\right) & (5.10) \\ \quad \text{if } \textit{at least} \text{ one of } F, G \in \mathscr{D}_0 \textit{ and } F \cap G = \varnothing & \\ \langle F, G\rangle = g^{jk}(-1)^{(F)+1}\dfrac{\partial F}{\partial s_j}\dfrac{\partial G}{\partial s_k} & (5.11) \\ \quad \text{if at least one of } F, G \text{ is } \in \mathscr{D}_0 \text{ and } F \cap G \neq \varnothing & \\ [F, G] = FG - GF \qquad \text{if} \quad F, G \in \mathscr{D}_1 & (5.12) \end{cases}$$

The signature of g^{jk} is specified in the particular cases which follow and the summation convention applies. Then it can be shown that $\{,\}_{\mathscr{D}}$ is a Poisson bracket satisfying properties (2.1)-(2.4). It has been shown (Sherry, 1989*a-c*) that the bracket $\langle,\rangle$ is a *super-Poisson bracket* with the following properties:

(i) It is bilinear.
(ii) $\langle f, g\rangle = -(-1)^{(f)(g)}\langle g, f\rangle$.
(iii) $\langle f, gh\rangle = \langle F, g\rangle h + (-1)^{(f)(g)} g\langle f, h\rangle$.
(iv) $(-1)^{(f)(h)}\langle f, \langle g, h\rangle\rangle + (-1)^{(g)(f)}\langle g, \langle h, f\rangle\rangle + (-1)^{(h)(g)}\langle h, \langle f, g\rangle\rangle = 0$, where $f, g, h \in \mathscr{G}$.

In accordance with definition (3.2), a Hamiltonian vector field V_H associated with an element $H \in \mathscr{D}$ satisfies

$$V_H(q_j) = \{q_j, H\} = \frac{\partial H}{\partial p_j}; \qquad V_H(p_j) = \{p_j, H\} = -\frac{\partial H}{\partial q_j}$$
$$V_H(s_j) = \{s_j, H\} = \frac{\partial H}{\partial s_j} \tag{5.13}$$

In order to formulate models of classical electrons, we need to introduce the notion of "classical spin" in analogy with the quantum concept. We follow Dirac (1958) in defining spin to be a variable s_j, $j = 1, 2, 3$, with the following properties:

(a) $\{S_j, S_k\} = S_l$, where j, k, l form a cyclic permutation of $1, 2, 3$.
(b) In the case of a free particle, or a particle in a central force field, angular momentum is not conserved, but the sum of angular momentum and spin is a conserved quantity.

In addition, we shall show that classical spin can be quantized to yield the equivalent of "quantum" spin 1/2.

We first consider the nonrelativistic case. All the elements of $\mathscr{D}$ are assumed to be functions of t and define $g^{jk} = \delta^{jk}$, $j, k = 1, 2, 3$. The elements of the Grassmann algebra $\mathscr{G}$ can be given a physical interpretation in the following way. The super-Poisson bracket of the generators s_j has the form

$$\langle s_j, s_k\rangle = \delta_{jk}$$

as well as being symmetric, so if the s_j are interpreted as *vectors* of a three-dimensional Euclidean space E_3, then this bracket satisfies the conditions of a metric; the s_j form an orthonormal basis of E_3 relative to this metric.

It has been shown (Sherry, 1989*c*) that bivectors of the type $S_j = -s_k s_l$, where J, k, l form a cyclic permutation of $1, 2, 3$, satisfy condition (a) of spin. In addition, the s_j transform like vectors under $SO(3)$ since

$$\{S_j, s_k\} = s_l \tag{5.14}$$

That the S_j also satisfy the condition (b) is shown by considering an analog of spin-orbit coupling of a particle in a central field with the relevant term in the Hamiltonian of the form

$$H = f(r) \sum_{j=1}^{3} S_j L_j \tag{5.15}$$

where r is a constant distance from the source of the field and $L_j = q_k p_l - q_l p_k$ are the equivalents of the components of the angular momentum in $\mathscr{D}$. It can be shown that $dL_j/dt \neq 0$, while $(d/dt)(L_j + S_j) = 0$.

In the relativistic case, g^{jk}, $j, k = 0, 1, 2, 3$, has signature 2, and we assume that all elements of $\mathscr{D}$ are functions of the parameter τ which represents the proper time. Let $S_\alpha = -s_\beta s_\gamma$, where α, β, γ form a cyclic permutation of $1, 2, 3$. In a similar fashion to the nonrelativistic case, the bracket $\langle s_j, s_k\rangle$ has the properties of the Minkowski metric, while

$$\{S_\alpha, S_\beta\} = S_\gamma \tag{5.16}$$

so that the S_α satisfy condition (a) of the classical spin. The s_j can be interpreted as an orthonormal basis relative to g^{jk}.

The Hamiltonian for the free relativistic classical electron is chosen to be

$$H = (mc)^{-1} g^{jk} p_i s_k \tag{5.17}$$

and it has been shown (Sherry, 1989*a*) that if $L_{\alpha\beta} = q_\alpha p_\beta - q_\beta p_\alpha$ represents the algebraic equivalent of the angular momentum tensor, then $(d/d\tau)$-$(L_{\alpha\beta}) \neq 0$, while $(d/dt)(L_{\alpha\beta} + S_\gamma) = 0$, so that condition (b) is also satisfied.

6. ALGEBRAIC MODELS OF ELECTRONS

The Poisson algebra in the quantum case is $(Q, \{,\}_Q)$ where Q is the Clifford–Weyl algebra. Q is the direct product

$$Q = \mathscr{W} \otimes \mathscr{C} \tag{6.1}$$

where $\mathscr{W}$ is an nth Weyl algebra and $\mathscr{C}$ is an n-dimensional Clifford algebra. Q is an associative algebra defined over $\mathbb{C}$ with generators $\tilde{x}_j = (\tilde{q}_j, \tilde{p}_j, \tilde{s}_j)$, where $\tilde{s}_j$ are the generators of $\mathscr{C}$, satisfying

$$[\tilde{q}_j, \tilde{q}_k] = [\tilde{p}_j, \tilde{p}_k] = 0 = [\tilde{s}_j, \tilde{q}_k] = [\tilde{s}_j, \tilde{p}_k] \tag{6.2}$$

$$[\tilde{q}_j, \tilde{p}_k] = g^{jk} = \tfrac{1}{2}(\tilde{s}_j\tilde{s}_k + \tilde{s}_k\tilde{s}_j) \tag{6.3}$$

The Poisson bracket on Q is the commutator $[,]$, so that if $\tilde{F}, \tilde{G} \in Q$, then

$$\{\tilde{F}, \tilde{G}\}_Q = [\tilde{F}, \tilde{G}] \tag{6.4}$$

A Hamiltonian derivation $V_{\tilde{H}}$ associated with $\tilde{H} \in Q$ satisfies

$$V_{\tilde{H}}(\tilde{q}_j) = \{\tilde{q}_j, \tilde{H}\}_Q = \frac{\partial \tilde{H}}{\partial \tilde{p}_j} \tag{6.5}$$

$$V_{\tilde{H}}(\tilde{p}_j) = \{\tilde{p}_j, \tilde{H}\}_Q = -\frac{\partial \tilde{H}}{\partial \tilde{q}_j} \tag{6.6}$$

$$V_{\tilde{H}}(\tilde{s}_j) = \{\tilde{s}_j, \tilde{H}\}_Q = \frac{\partial \tilde{H}}{\partial \tilde{s}_j} \tag{6.7}$$

and these also serve to define the derivations $\partial/\partial\tilde{q}_j$, $\partial/\partial\tilde{p}_j$, and $\partial/\partial\tilde{s}_j$.

In the case of the nonrelativistic electron, Q is generated by $\tilde{q}_j$, $\tilde{p}_j$, $\tilde{s}_j$, where $j = 1, 2, 3$ and $g^{jk} = \delta^{jk}$. If the $\tilde{s}_j$ are interpreted as an orthonormal basis of E_3, then $\mathscr{C}$ is the complex Pauli algebra and the $\tilde{s}_j$ can be represented by the Pauli matrices. All elements of Q are assumed to be functions of t and a Hamiltonian structure is defined by a Hamiltonian $\tilde{H} \in Q$ satisfying

$$\frac{d\tilde{q}_j}{dt} = \{\tilde{q}, \tilde{H}\}_Q, \qquad \frac{d\tilde{p}_j}{dt} = \{\tilde{p}_j, \tilde{H}\}_Q, \qquad \frac{d\tilde{s}_j}{dt} = \{\tilde{s}_j, \tilde{H}\}_Q \tag{6.8}$$

Spin of 1/2 is defined to be an element $\tilde{S}_j \in Q$ satisfying properties (a) and (b) of Section 5. Suitable candidates for the spin variables are bivectors $\tilde{S}_j = \frac{1}{2}\tilde{s}_l\tilde{s}_k$, where s, k, l form a cyclic permutation of 1, 2, 3. First,

$$\{\tilde{S}_j, \tilde{S}_k\}_Q = \tfrac{1}{4}[\tilde{s}_l\tilde{s}_k, \tilde{s}_j\tilde{s}_l] = \tfrac{1}{2}\tilde{s}_k\tilde{s}_j = \tilde{S}_l \tag{6.9}$$

thus satisfying condition (a).

Note that

$$\{\tilde{S}_j, \tilde{s}_k\} = \tfrac{1}{2}[\tilde{s}_l\tilde{s}_k, \tilde{s}_k] = \tilde{s}_l$$

so that $\tilde{s}_l$ transform like vectors under $SO(3)$.

Second, we consider the quantum version of the spin-orbit coupling Hamiltonian of the electron in a central field of (5.15) given by

$$\tilde{H} = f(r) \sum_{j=1}^{3} \tilde{S}_j\tilde{L}_j$$

Here $\tilde{L}_j$ is the jth component of the orbital angular momentum. Then, just as in the classical case, we obtain

$$\begin{aligned}\frac{d\tilde{L}_j}{dt} &= [\tilde{L}_j, \tilde{H}] \\ &= f(r)(\tilde{S}_k\tilde{L}_l - \tilde{S}_l\tilde{L}_k) \neq 0\end{aligned}$$

where j, k, l form a cyclic permutation of 1, 2, 3, while

$$\frac{d\tilde{S}_j}{dt} = f(r)(\tilde{L}_k\tilde{S}_l - \tilde{L}_l\tilde{S}_k) \neq 0$$

but where

$$\frac{d\tilde{L}_j}{dt}+\frac{d\tilde{S}_j}{dt}=0$$

thus satisfying condition (b).

In the relativistic case, g^{jk} ($j, k=0, 1, 2, 3$) has signature 2 and the parameter is the proper time τ. The quantized version of the Hamiltonian of (5.20) becomes $\tilde{H}=(g^{jk}/mc)p_j\tilde{s}_k$, representing the free electron.

The angular momentum is given by $\tilde{L}_\alpha=\tilde{q}_\beta\tilde{p}_\gamma-\tilde{q}_\gamma\tilde{p}_\beta$, while spin has components $\tilde{S}_\alpha=\frac{1}{2}\tilde{s}_\beta\tilde{s}_\gamma$, where α, β, γ form a cyclic permutation of 1, 2, 3.

First, as in the nonrelativistic case,

$$\{\tilde{S}_\alpha, \tilde{S}_\beta\}_Q=\tilde{S}_\gamma$$

so that the condition (a) of spin is satisfied. Then

$$\frac{d\tilde{L}_\alpha}{d\tau}=\{\tilde{L}_\alpha, \tilde{H}\}_Q=\frac{1}{mc}(\tilde{p}_\gamma\tilde{s}_\beta-\hat{p}_\beta\tilde{s}_\gamma)\neq 0$$

while

$$\frac{d\tilde{S}_\alpha}{d\tau}=\{\tilde{S}_\alpha, \tilde{H}\}$$

$$=\frac{1}{mc}(\tilde{p}_\beta\tilde{s}_\gamma-\tilde{p}_\gamma\tilde{s}_\beta)$$

so that $(d/d\tau)(\tilde{L}_\alpha+\tilde{S}_\alpha)=0$ and condition (b) is satisfied.

It was shown (Sherry, 1989*a*) that in the particular realization given by

$$\tilde{s}_j\to\gamma_j$$
$$\tilde{q}_j\to\hat{q}_j$$
$$\tilde{p}_j\to\hat{p}_j=\frac{\hbar}{i}\frac{\partial}{\partial\hat{q}^j}$$

where $\hat{q}_j$, $\hat{p}_j$, and γ_j are the position, momentum, and Dirac γ-operators, the Hamiltonian $\tilde{H}=(g^{jk}/mc)\hat{p}_j\gamma_k$ can be transformed into

$$\hat{H}_{\mathrm{D}}=c\gamma^0\gamma^\mu p_\mu+mc^2\gamma^0$$

which is the Hamiltonian in the covariant form of Dirac's theory of the electron.

Quantization is defined to be the map ψ given by

$$\psi\colon\quad \mathscr{D}\to Q$$
$$\psi\colon\quad q_j\to\tilde{q}_j,\quad p_j\to\tilde{p}_j=\frac{\partial}{\partial\tilde{q}_j};\quad s_j\to\tilde{s}_j=s_j+g^{jk}\frac{\partial}{\partial s_k}$$

and it is straightforward to show that these elements satisfy the relations (6.2) and (6.3).

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